New $AdS_4 \times X_7$ geometries with $\mathcal{N} = 6$ in M theory

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ABSTRACT: We study supersymmetric $AdS_4 \times X_7$ solutions of 11-dim supergravity where the tri-Sasakian space X_7 has generically $\mathrm{U}(1)^2 \times \mathrm{SU}(2)_R$ isometry. The compact and regular 7-dim space $X_7 = S(t_1, t_2, t_3)$ is originated from 8-dim hyperkahler quotient of a 12-dim flat hyperkahler space by $\mathrm{U}(1)$ and belongs to the class of the Eschenburg space. We calculate the volume of X_7 and that of supersymmetric five cycle via cohomological localization. From this we discuss the 3-dim dual superconformal field theories with $\mathcal{N} = 3$ supersymmetry.

Contents

1. Introduction and conclusion

The AdS-CFT correspondence predicts that the type IIB-theory on the supergravity solution $AdS_5 \times S^5$ with appropriate 5-form field strength is dual to a $\mathcal{N} = 4$ supersymmetric 4-dim SU(N) gauge theory, which is a superconformal field theory [1]. When the space S^5 is replaced by a 5-dim Sasaki Einstein space, the dual gauge theory has less supersymmetry with more complicated group and matter structure [2]. This conformal theory can be regarded as a field theory on a stack of D3 branes sitting at the singular tip of a Ricci flat 6-dim cone whose base is the Sasaki-Einstein space [3–5].

The AdS-CFT correspondence for the Freund-Rubin form, $AdS_4 \times X_7$, of a supersymmetric solution of 11-dim supergravity implies that the M-theory on such background is dual to a supersymmetric 3-dim superconformal field theory. Again the field theory arises as the SCFT on a stack of M2 branes at the singular apex of 8-dim Ricci-flat space with special holonomy, whose base is 7-dim X_7. One has to have at least one Killing spinor on X_7 to have a special holonomy. Recently there have been found several countable series of Sasaki-Einstein spaces in 5-dim and 7-dim, and their AdS-CFT correspondence has been studied [6–8]. Especially the corresponding 3-dim SCFT's have $\mathcal{N} = 1$, or 2 supersymmetries.

In this work we focus on a class of countable series of the $AdS_4 \times X_7$ spaces, which has not been studied before. The dual SCFT has $\mathcal{N} = 3$ and is the SCFT on the M2 branes on the singular tip of the 8-dim hyperkahler cone with Sp(2) holonomy and with its base being a tri-Sasakian space X_7. The 8-dim hyperkahler cone is obtained by a hyperkahler quotient of $\mathbb{R}^{12}$ by a U(1) symmetry group [9]. (In general, one could have started from flat $\mathbb{R}^{4n+8}$ space with U(1)^n hyperkahler quotient with $n \geq 1$ but here we restrict to $n = 1$ case for simplicity.) Our tri-Sasakian space $X_7(\mathbf{t}) = S(t_1, t_2, t_3)$ is characterized by three natural

numbers t_1, t_2, t_3 and has $\mathrm{SU}(2)_R \times \mathrm{U}(1)^2$, $\mathrm{SU}(2)_R \times \mathrm{SU}(2) \times \mathrm{U}(1)$ or $\mathrm{SU}(2)_R \times \mathrm{SU}(3)$ isometry depending on none, two, or all of t_a coincide, respectively. We calculate the volume of the X_7 and its supersymmetric 5-cycles Σ_5 and obtain a rational expression for the ratio of the volume of X_7 and that of the unit 7-sphere and so on. We also discuss the dual 3-dim superconformal field theories with $\mathcal{N} = 3$. During our investigation, we found this type of space has appeared before in the mathematics literature [10 – 12] where it is known as the Eschenburg space [13]. However, our calculation of the volumes of the space and the super 5-cycles found in this paper seems original.

The simplest case with $X_7 = S(1, 1, 1)$ is known as $N(1, 1)$ and its cone is the relative moduli space of a single instanton in $\mathrm{SU}(3)$ gauge group, which is hyperkahler 8-dim space with one scale and the coset space $\mathrm{SU}(3)/\mathrm{U}(1)$. There has been considerable work on the AdS-CFT correspondence on the $AdS_4 \times N(1, 1)$ space [14 – 16]. Especially as $N(1, 1)$ is homogeneous, one can study the Kaluza-Klein modes of the theory to compare them with the dual SCFT. More recent investigation on $AdS_4 \times X_7$ space with known X_7 and its marginal deformation can be found in ref. [17].

Our work has been motivated in part by the effort to understand the mysterious 3-dim $\mathcal{N} = 16$ supersymmetric conformal field theory which is the low energy theory of N parallel M2 branes. It can be regarded as the strong coupling limit $e^2 \to \infty$ of $\mathcal{N} = 16$ supersymmetric Yang-Mills theory, as one can easily see in the M-theory limit of D2 branes. One may deform the $\mathcal{N} = 16$ supersymmetric Yang-Mills by adding Chern-Simons terms, so that the resulting theory has a less supersymmetry $\mathcal{N} = 3$ [18, 19]. In the infrared limit or strong coupling limit $e^2 \to \infty$, the theory becomes purely Chern-Simons Higgs, which is superconformal. While the Chern-Simons level k is quantized to be integer, the small $|k|$ limit is the strong-coupling limit. Unfortunately its physics is not well understood. One may still hope that the physics near $k = 0$ is similar to that of $N = 16$ superconformal theory. (See for a similar idea in ref. [20].)

Another motivation was to try to understand further the old result on the superconformal field theory dual for the AdS geometry with tri-Sasakian space $X_7 = N(1, 1)$ whose 8-dim cone is the relative moduli space of a single instanton in $\mathrm{SU}(3)$ gauge theory. While the corresponding field theory may have some component of Chern-Simons theory, the t'Hooft coupling from the geometry seems to be related to the parameter N of the corresponding gauge group $\mathrm{SU}(N) \times \mathrm{SU}(N)$, instead of the generic 'tHooft coupling N/k of the Chern-Simons-Higgs theory where k is the integer quantized Chern-Simons level. Our models would provide more examples along this line.

Final motivation was to try to construct new tri-Sasakian geometry similar to instanton moduli space by generalizing the moduli space of three distinct magnetic monopoles, which could be constituent of a single instanton of $\mathrm{SU}(3)$ theory in $R^3 \times S^1$ geometry [21]. Thus one wants to generalize the moduli space of N distinct magnetic monopoles in $\mathrm{SU}(N)$ theory broken to $\mathrm{U}(1)^{N-1}$ in $R^3 \times S^1$. There would be magnetic monopoles for each simple root of the extended Dynkin diagram of $\mathrm{SU}(N)$ gauge group. But we generalize the interaction strength between nearest neighbor by arbitrary magnitude. Only requirement is that the geometry is smooth whenever magnetic monopoles are coming together except when all of them are coming together. In appendix this is shown to lead to the interaction strength

between each link to be some natural number instead of the unity as in the $SU(N)$ case. In the limit where $N - 1$ monopoles become massless, the relative geometry has only one scale parameter which controls the overall size of the system, and so a cone-like geometry with a singularity at the apex of the cone. Our geometry X_7 can also obtain from this approach.

The geometry we are interested in is the $AdS_4 \times X_7$ type solutions of $D = 11$ supergravity where X_7 is an Einstein manifold and the four-form field strength $F_4 \sim vol_{AdS_4}$. We normalize the metric on X_7 so that $R_{\mu\nu}(X_7) = 6g_{\mu\nu}(X_7)$. To preserve some of 32 supersymmetry of 11-dim supergravity, the eight dimensional cone $\mathcal{M}_8$ over X_7 with the metric

$$ds^2_{\mathcal{M}_8} = dr^2 + r^2 ds^2(X_7) \tag{1.1}$$

should be Ricci-flat and have special holonomy. (For example see ref. [22].) When the cone has $Sp(2) = SO(5)$ holonomy, and so hyper-Kähler, the space X_7 is tri-Sasakian and the dual theory is a $\mathcal{N} = 3$ SCFT.

Such a geometry arises as the near horizon limit of $M2$ branes lying at the singular apex of the Ricci-flat cone $\mathcal{M}_8$. The dual SCFT lives on the $M2$ branes. The flux of F_4 on X_7 is proportional to the number of $M2$ branes. Baryonic states of SCFT are dual to five-branes wrapping five-cycles Σ_5 in the manifold X_7. For supersymmetirc states the five-cycles must lift to supersymmetric 6-cycles in the cone $\mathcal{M}_8$. A supersymmetric 6-cycle will be holomorphic with respect to one of the three complex structures, breaking two of six supersymmetries. The dimension of the baryonic operators are given by the geometric formula [15]

$$\Delta = \frac{\pi N}{6} \frac{\mathrm{Vol}(\Sigma_5)}{\mathrm{Vol}(X_7)}. \tag{1.2}$$

In SCFT, they can often be predicted from their R-charges. Comparing the two predictions is then a non-trivial check for the gauge/gravity correspondence.

This Eschenburg space $X_7 = S(t_1, t_2, t_3)$ [10] can be regarded as a left-quotient space of $SU(3)$ group manifold by $U(1)$ group whose elements are $\mathrm{diag}(e^{it_1\psi}, e^{it_2\psi}, e^{it_3\psi})$. Their homological properties seem to be known. Here we provide an explicit metric and calculation of the volume of the space and supersymmetric 5-cycles. The space $X_7 = S(t_1, t_2, t_3)$ depends only on the t_a's up to overall common factor. It is homogeneous with $SU(3) \times SU(2)_R$ symmetry when $t_1 = t_2 = t_3$. When $t_1 = t_2 \neq t_3$, the space has co-homogeneity one with $SU(2) \times U(1) \times SU(2)_R$ symmetry. When all t_a are different from each other, the space has co-homogeneity two with $U(1)^2 \times SU(2)_R$ symmetry. We find the two kinds of expressions for the metric for $X_7(\mathbf{t})$. The first one is explicit but not useful. The second one is more implicit but shows the symmetric and cone structures clearly.

Instead of using the metric, we use the equivariant cohomology and localization technique [23] to calculate the volume $X_7(\mathbf{t})$. This approach is somewhat esoteric and so the detail is provided here. We find that the ratio of the volume of the tri-Sasakian space X_7 and that for any supersymmetric 5-dim cycle $\Sigma_5(\mathbf{t})$ is independent of $\mathbf{t} = (t_1, t_2, t_3)$ and identical to the ratio for the volume of unit 7-sphere and that of unit 5-sphere. The explicit

form for the volumes are

$$\frac{vol(X_7(\mathbf{t}))}{vol(S_7)} = \frac{vol(\Sigma_5(\mathbf{t}))}{vol(S_5)} = \frac{t_1 t_2 t_3 (t_1 t_2 + t_2 t_3 + t_3 t_1)}{l.c.m.(t_1 t_2, t_2 t_3, t_3 t_1)(t_1 + t_2)(t_2 + t_3)(t_3 + t_1)}, \qquad (1.3)$$

where $l.c.m.$ means the least common multiple and $vol(S_7) = \pi^4/3$ for the unit 7 sphere and $vol(S_5) = \pi^3$ for the unit 5 sphere. The maximum of the above ratio for any $\mathbf{t}$ appears when $t_1 = t_2 = t_3 = 1$ for the well-known space $N(1,1)$.

The dual superconformal field theory in three dimension is $SU(N)_1 \times SU(N)_2$ gauge theory with $\mathcal{N} = 3$ supersymmetry. The matter fields $U_a = (u_a, -v_a^*)$ are hypermultiplets in $\mathcal{N} = 4$ language belonging to the symmetrized product representation $Sym^{t_a}(N)$ of the fundamental representation of $SU(N)_1$ and that $Sym^{t_a}(\bar{N})$ of the anti-fundamental representation of $SU(N)_2$. The internal global symmetry is again $SU(2)_R \times U(1)^2$ for distinct t_a. The chiral primary operators and baryonic operators show that one can assign a chiral dimension $1/2$ for the U_a field and its complex conjugates.

There are several directions to pursue. Our SCFT is again mysterious as the $\mathcal{N} = 6$ superconformal field theory in 3-dim as it is not quite the Chern-Simons theory. They may be the strong-coupling limit of the supersymmetric Yang-Mills Chern-Simons theory with $\kappa \to 0$. One curious aspect of our AdS-CFT correspondence is that there is no obvious geometry for the dual theory when $t_1 = t_2 = t_3 \neq 1$ as the $X_7(\mathbf{t})$ is defined only up to common factors of t_a.

The plan of the paper is as follows. In section 2 we define the 8-dim hyperkahler space $\mathcal{M}_8(\mathbf{t})$ which is a singular cone and is obtained from a hyperkahler quotient of 3-dim quaternion space $\mathbb{H}^3 = \mathbb{R}^{12}$ by using a single $U(1)$ group. We find its metric explicitly and also show the space has the cone geometry. We identify its isometry. In section 3 we review the homology property of tri-Sasakian space $X_7(\mathbf{t})$ first. Then we calculate the volumes of $X_7(\mathbf{t})$ and supersymmetric 5-cycles Σ_5 in X_7 in the language of equivariant cohomology. In section 4, we identify the dual SCFT and study its properties. In appendix, we generalize the caloron moduli space.

2. Hyperkahler space $\mathcal{M}_8(\mathbf{t})$

Let us start from 12-dim flat hyperkahler space $\mathbb{H}^3$ defined by the three quaternions q_1, q_2, q_3. (See for example ref. [24] for an introduction.) Each quaternion is defined as

$$q_a = q_a^4 + i\boldsymbol{\sigma} \cdot \mathbf{q}_a, \quad \bar{q}_a = q_a^4 - i\boldsymbol{\sigma} \cdot \mathbf{q}_a, \qquad (2.1)$$

with four real numbers q_a^μ, $\mu = 1, 2, 3, 4$ and three Pauli matrices $\sigma^1, \sigma^2, \sigma^3$. The Euclidean flat metric on 12-dim is

$$ds^2 = \sum_a \frac{1}{2} \mathrm{tr}(dq_a \otimes d\bar{q}_a) = \sum_a dq_a^\mu dq_a^\mu, \qquad (2.2)$$

and the three Kähler forms are

$$\boldsymbol{\omega} \cdot \boldsymbol{\sigma} = \frac{1}{2} dq \wedge d\bar{q}. \qquad (2.3)$$

Sometimes we use complex coordinates for quaternions as

$$q_a = \begin{pmatrix} u_a & v_a \\ -\bar{v}_a & \bar{u}_a \end{pmatrix}, \tag{2.4}$$

in which the metric becomes

$$ds^2 = \frac{1}{2} \sum_a (du_a \otimes d\bar{u}_a + dv_a \otimes d\bar{v}_a + c.c.) . \tag{2.5}$$

Another useful coordinate for quaternions is

$$q_a = p_a e^{i\sigma_3 \psi_a}, \tag{2.6}$$

where p_a is pure imaginary, or $\bar{p}_a = -p_a$. In terms of the 3-dim Cartesian coordinates $\mathbf{r}_a$ such that

$$i\mathbf{r}_a \cdot \boldsymbol{\sigma} = q i\sigma_3 \bar{q} = -ip_a \sigma^3 p_a \tag{2.7}$$

and angle variable ψ_a, the flat metric on 12-dim becomes

$$ds^2 = \frac{1}{4} \sum_a \left(\frac{d\mathbf{r}_a^2}{r_a} + r_a (d\psi_a + \mathbf{w}_a \cdot d\mathbf{r}_a)^2 \right), \tag{2.8}$$

where $r_a = |\mathbf{r}_a|$ and $\nabla \times \mathbf{w}_a(\mathbf{r}_a) = \nabla(1/r_a)$.

For each triple natural numbers t_1, t_2, t_3, we consider a corresponding abelian symmetry $U_{\mathbf{t}}(1)$, under which

$$q_a \rightarrow q_a e^{i\sigma_3 t_a \chi}, \quad a = 1, 2, 3 . \tag{2.9}$$

The $U_{\mathbf{t}}(1)$ is unique up to a common factor on triples. The corresponding moment map $\boldsymbol{\mu}$ is

$$\begin{aligned} \boldsymbol{\mu} \cdot \boldsymbol{\sigma} &= \sum_a t_a q_a \sigma_3 \bar{q}_a = \sum_a t_a \mathbf{r}_a \\ &= \sum_a t_a \begin{pmatrix} |u_a|^2 - |v_a|^2 & -2u_a v_a \\ -2\bar{u}_a \bar{v}_a & -|u_a|^2 + |v_a|^2 \end{pmatrix} . \end{aligned} \tag{2.10}$$

The space which satisfies the constraint $\boldsymbol{\mu} = 0$ becomes 9-dimensional. Once we mod out $U(1)_t$ on this space, the resulting quotient space becomes 8-dim hyperkahler space,

$$\mathcal{M}_8(\mathbf{t}) = \boldsymbol{\mu}^{-1}(0)/U_{\mathbf{t}}(1) . \tag{2.11}$$

This process of hyperkahler quotient is defined by three natural numbers t_1, t_2, t_3. As $\mathcal{M}_8(\mathbf{t})$ is hyperkahler, it is Ricci-flat automatically.

Let us consider in detail the symmetry of the hyperkahler space $\mathcal{M}_8(\mathbf{t})$. The first one is the $SU(2)_R$ symmetry which rotates three complex structures in 12-dim space,

$$q_a \rightarrow \exp(-\frac{i}{2}\boldsymbol{\epsilon} \cdot \boldsymbol{\sigma})q_a, \ a = 1, 2, 3 , \tag{2.12}$$

where ϵ are the SU(2) parameters. Under this SU(2) transformation, $\mathbf{r}_a$ for each a transforms as a vector. This is commuting with the hyperKähler quotient and so the resulting space has SU(2)$_R$ symmetry. The additional symmetry arises from the transformation

$$q_a \to q_b \left(\exp iT\sigma^3\right)_{ba} \tag{2.13}$$

where T is the U(3) generator which commutes with the U(1)$_{\mathbf{t}}$ generator

$$\mathbf{t} = \mathrm{diag}(t_1, t_2, t_3), \tag{2.14}$$

and leaves $U_{\mathbf{t}}(1)$ invariant subspace intact. Thus when $t_1 = t_2 = t_3$, the resulting symmetry is SU(3). When $t_1 = t_2 \neq t_3$, the resulting symmetry is SU(2) $\times$ U(1). When t_1, t_2, t_3 are all different, the resulting symmetry would be U(1)^2.

For $t_1 = t_2 = t_3$, the resulting 8-dim space is the moduli space of a single SU(3) instanton in the center of mass frame. The 8 parameters denote a single scale parameter and 7 coordinates for the coset space SU(3)/ U(1), and so the space is cone-like. For generic t_a, the metric is more complicated. A simple way to write the metric is to start from the flat metric (2.8) and express the $\mathbf{r}_3$ in terms of $\mathbf{r}_1$ and $\mathbf{r}_2$ by using the moment map (2.10) so that with $A = 1, 2$

$$4ds^2 = C_{AB}d\mathbf{r}_A d\mathbf{r}_B + C^{AB}(d\psi_A + \mathbf{w}_{AC} \cdot d\mathbf{r}_C)(d\psi_B + \mathbf{w}_{BD} \cdot d\mathbf{r}_D), \tag{2.15}$$

where

$$
\begin{aligned}
C_{11} &= \frac{1}{r_1} + \frac{t_1^2}{t_3|t_1\mathbf{r}_1 + t_2\mathbf{r}_2|}, \\
C_{22} &= \frac{1}{r_2} + \frac{t_2^2}{t_3|t_1\mathbf{r}_1 + t_2\mathbf{r}_2|}, \\
C_{12} &= C_{21} = \frac{t_1 t_2}{t_3|t_1\mathbf{r}_1 + t_2\mathbf{r}_2|},
\end{aligned}
\tag{2.16}
$$

and the vector potential satisfies $\nabla_C \times \mathbf{w}_{AB} = \nabla_C C_{AB}$. This metric is hyperkahler and regular unless $r_1 = r_2 = 0$ simultaneously [25, 26].

To express the metric so that the cone-structure is manifest needs more work. Let us focus on the generic case where all t_a are different. The moment map vanishes and so $t_1\mathbf{r}_1 + t_2\mathbf{r_2} + t_3\mathbf{r}_3 = 0$, which defines a triangle whose side length are $t_1 r_1, t_2 r_2, t_3 r_3$. Using the SU(2)$_R$ transformation, we can put this triangle on the 1-2 plane. In this case, the complex coordinates u_a and v_a satisfy

$$|u_a| = |v_a| = \sqrt{\frac{r_a}{2}}, \quad \sum_{a=1}^{3} t_a u_a v_a = 0. \tag{2.17}$$

The general configuration would be made of the rotation of the triangle in space and also the phase rotation of u_a and v_a in opposite way. Triangle has three independent parameters. The spatial rotation has three independent parameters. The relative phase of u_a, v_a variables has three independent parameters, one of which is the global U(1) which should be mod out. Thus there are eight independent parameters.

To specify the moduli parameter for the triangle, we choose the parameters to be $u_a = v_a = \sqrt{r_a/2} = |u_a|e^{i\varphi_a/2}$, which implies the moduli metric of the triangle to be

$$ds_\Delta^2 = \frac{1}{4}\sum_{a=1}^{3}\left(\frac{1}{r_a}dr_a^2 + r_a d\varphi_a^2\right),\tag{2.18}$$

with the condition (2.17) being

$$\sum_a t_a r_a e^{i\varphi_a} = 0.\tag{2.19}$$

This is the condition for three complex vectors $t_a r_a e^{i\varphi_a}$ to form a triangle. This constraint depends only on the relative angles θ_a of vectors as

$$\theta_1 = \varphi_3 - \varphi_2,\qquad \theta_2 = 2\pi + \varphi_1 - \varphi_3,\qquad \theta_3 = \varphi_2 - \varphi_1,\tag{2.20}$$

where $0 \le \varphi_a < 2\pi$. Only two of the relative angles are independent as $\theta_1 + \theta_2 + \theta_3 = 2\pi$. The overall orientation angle $\varphi = \frac{\varphi_1+\varphi_2+\varphi_3}{3}$ of the triangle is a part of the rotational degrees from $SU(2)_R$. The above triangle condition (2.19) implies the three following conditions on the length and relative angles as given in elementary geometry:

$$\begin{aligned}
t_1^2 r_1^2 &= t_2^2 r_2^2 + t_3^2 r_3^2 + 2t_2 r_2 t_3 r_3 \cos\theta_1\ ,\\
t_2^2 r_2^2 &= t_3^2 r_3^2 + t_1^2 r_1^2 + 2t_3 r_3 t_1 r_1 \cos\theta_2\ ,\\
t_3^2 r_3^2 &= t_1^2 r_1^2 + t_2^2 r_2^2 + 2t_1 r_1 t_2 r_2 \cos\theta_3\ ,
\end{aligned}\tag{2.21}$$

of which only two are independent. Thus these conditions reduce the independent variables to three, which we choose as one length variable, and two relative angle variables.

To solve the above constraints (2.21), let us introduce an angle variable A, a length square variable L, and an area variable S such that

$$\begin{aligned}
A &= -(\cot\theta_1 + \cot\theta_2 + \cot\theta_3)\ ,\\
L &= \sum_a t_a^2 r_a^2\ ,\\
S &= t_1 r_1 t_2 r_2 \sin\theta_3 = t_3 r_3 t_1 r_1 \sin\theta_2 = t_2 r_2 t_3 r_3 \sin\theta_1
\end{aligned}\tag{2.22}$$

Note that S is twice the area of the triangle, and the triangle condition implies that

$$S = \frac{L}{2A}\quad,\quad t_a^2 r_a^2 = L\left(1 + \frac{\cot\theta_a}{A}\right)\quad\text{for each } a\tag{2.23}$$

with $A \ge 0$. Let us now introduce the radial variable in 12-dim flat space as the length variable,

$$r = \sqrt{|q_1|^2 + |q_2|^2 + |q_3|^2} = \sqrt{r_1 + r_2 + r_3}\,.\tag{2.24}$$

Defining a function B of angles θ_a as

$$B = \sum_a \frac{1}{t_a}\sqrt{\left(1 - \frac{\cot\theta_a}{\cot\theta_1 + \cot\theta_2 + \cot\theta_3}\right)}\,,\tag{2.25}$$

we see the length variable L is given in terms of three independent variables r, θ_a as follows,

$$L = \frac{r^4}{B^2} \,. \tag{2.26}$$

So the variables r_a can be written in terms of scale variable r and angles θ_a as follows,

$$r_a = r^2 \rho_a \,, \qquad \rho_a \equiv \frac{1}{t_a} \sqrt{\left(1 - \frac{\cot \theta_a}{\sum_b \cot \theta_b} \right)} \Bigg/ \sum_c \frac{1}{t_c} \sqrt{\left(1 - \frac{\cot \theta_c}{\sum_d \cot \theta_d} \right)}. \tag{2.27}$$

Note that three functions ρ_a of angles θ_a satisfies the condition $\rho_1 + \rho_2 + \rho_3 = 1$.

The moduli space of the triangle on the plane would be then

$$ds_\Delta^2 = \frac{1}{4} \sum_a \left(\frac{1}{r_a} dr_a^2 + r_a d\varphi_a^2 \right) = dr^2 + \frac{r^2}{4} \sum_a \left(\frac{d\rho_a^2}{\rho_a} + d\varphi_a^2 \right), \tag{2.28}$$

as $\sum \rho_a = 1$. Note the φ_a can be written as the relative angles θ_a and the overall orientation of the triangle on the plane. The metric for the $\mathcal{M}_8$ can be now obtained by parameterizing quaternion as follows

$$Q \equiv \begin{pmatrix} u_1 & u_2 & u_3 \\ -\bar{v}_1 & -\bar{v}_2 & -\bar{v}_3 \end{pmatrix} = R Q_0 T \,, \tag{2.29}$$

where the R is an SU(2) element parameterized by Euler angle, which includes the orientation of the triangle on the plane, T is a diagonal U(3) element, say, $T = \text{diag}(e^{i\psi_1}, e^{i\psi_2}, e^{i\psi_3})$ and Q_0 is the value of Q when the triangle is on 1-2 plane and so

$$Q_0 = \begin{pmatrix} \sqrt{r_1} e^{i\varphi_1/2} & \sqrt{r_2} e^{i\varphi_2/2} & \sqrt{r_3} e^{i\varphi_3/2} \\ -\sqrt{r_1} e^{-i\varphi_1/2} & -\sqrt{r_2} e^{-i\varphi_2/2} & -\sqrt{r_3} e^{-i\varphi_3/2} \end{pmatrix}. \tag{2.30}$$

The metric of the triangle is $ds_\Delta^2 = dQ_0 d\bar{Q}_0$ of the metric (2.28) and so the metric on the 9-dim space is

$$ds_{\boldsymbol{\mu}^{-1}(0)}^2 = \frac{1}{2} (dQ \otimes d\bar{Q} + d\bar{Q} \otimes dQ) \,. \tag{2.31}$$

It is trivial to mod out $U_{\mathbf{t}}(1)$ to get the 8-dim hyperkahler space $\mathcal{M}_8(\mathbf{t})$. The 3 Kähler forms are again given by

$$\boldsymbol{\omega} \cdot \boldsymbol{\sigma} = dQ \wedge d\bar{Q} \,. \tag{2.32}$$

The isometry of $\mathcal{M}_8(\mathbf{t})$ can be easily read. First of all the SU(2) transformation by R matrix leads to SU(2)$_R$ symmetry which mixes three complex structure. In addition there are U(1) $\times$ U(1) isometries from the tranformations given by T matrix modulo $U_{\mathbf{t}}(1)$, which are tri-holomorphic as they leave three kahler structures invariant. When some of t_1, t_2, t_3 become identical, these tri-holomoprhic isometries get enhanced. If only two of three are identical, U(1)^2 gets enhanced to U(1) $\times$ SU(2). If all three t_a are identical, U(1)^2 gets enhanced to SU(3).

The metric (2.31) has no mixed terms for dr and other angle variables including dR, dT as R and T are unitary. Thus, the metric on the hyperKähler space $\mathcal{M}_8$ has a cone structure,

$$ds_{\mathcal{M}_8}^2 = dr^2 + r^2 ds_{X_7}^2 \,. \tag{2.33}$$

It is also regular everywhere except at the tip of the cone. It is Ricci-flat and the 7-dim space X_7 is smooth everywhere without singularity and characterized by three natural numbers (t_1, t_2, t_3) without common factor. In the next section, we study properties of this 7-dim tri-Sasakian space $X_7(\mathbf{t})$.

3. Tri-Sasakian space $X_7(\mathbf{t})$

We have now the 8-dimensional hyperkahler space $\mathcal{M}_8(\mathbf{t})$, whose metric is cone-like and determined by three natural numbers t_1, t_2, t_3 modulo common factor. This space is smooth except at the tip of the cone. As the metric is written down almost explicitly, one has now the corresponding 7-dimensional tri-Sasakian space $X_7(\mathbf{t})$ which is smooth everywhere.

There are several properties of this space which are relevant for our consideration. The isometry of $X_7(\mathbf{t})$ is identical to the cone geometry $\mathcal{M}_8(\mathbf{t})$. With $t_1 = t_2 = t_3$, the unique space is equivalent to $S(1,1,1) = N(1,1)$. With $t_1 = t_2 \neq t_3$, there are class of geometry with $S(r,r,s)$ with coprime natural numbers r, s. Finally, when all t_a are different, we can assume that there is no common factor in them. This space $X_7(\mathbf{t}) = S(t_1, t_2, t_3)$ is non-singular. It is sometimes called the Eschenburg space. This toric Sasakian space is a subfamily of the more general spaces, bi-quotients of $U(3)$ group manifold, which was studied by Eschenburg [13].

The quotient of $S(t_1, t_2, t_3)$ by $SO(3)_R$ action is a quaternionic Kähler orbifold. For any $U(1)_R$ subgroup of $SO(3)_R$, one can locally write the metric as

$$ds^2(X_7) = (d\psi + \sigma)^2 + ds^2(M_6), \tag{3.1}$$

where M_6 is locally Kahler-Einstein. If the Reeb vector $\partial/\partial\psi$ has a closed orbit, then M_6 is in general Kahler orbifold. A tri-Sasakian space is regular if its quotient by $SO(3)_R$ is a quaternionic Kähler manifold. Our case $S(t_1, t_2, t_3)$ would be a smooth tri-Sasakian 7-manifold which is not regular unless $t_1 = t_2 = t_3$ in which case it is homogeneous. Indeed all homogeneous tri-Sasakian spaces in $4n + 3$ dimensions seem to be associated with Lie algebra and seem to be originated from the moduli space of a single instanton in a gauge theory of a given Lie group as it has only one scale parameter and is hyperkahler. The betti-numbers of the space $S(t_1, t_2, t_3)$ [10] are

$$b_0 = b_7 = 1, \quad b_1 = b_6 = 0, \quad b_2 = b_5 = 1, \quad b_3 = b_4 = 0, \tag{3.2}$$

which indicates one can have nontrivial wrapping of the geometry by $M2$ branes and M5 branes. The wrapping of 5 cycles by M5 branes leads to the baryonic objects in the dual SCFT.

One can obtain more general 7-dimensional toric tri-Sasakian space first proposed in ref. [11] by considering $N + 2$-quaternion space $q_a, a = 1, 2, \ldots, N + 2$ with N independent $U(1)_t$ groups acting on them with charge matrix

$$q_a \rightarrow q_a e^{i\sigma_3 t_a^A \chi_A}, \tag{3.3}$$

where $A = 1, 2, \ldots N$ leads to N abelian symmetry. There are N corresponding moment map

$$\mu^A = \sum_{a=1}^{N+2} t_a^A q_a \sigma_3 q_a \,. \tag{3.4}$$

The hyperKahler quotient of the $N + 2$ dimensional quaternion space by these abelian groups leads to 8-dimensional hyperkahler space. Since we propose dual SCFT only for $N = 1$ case in this work, generalization for $N \geq 2$ would be an interesting future problem.

3.1 The volume of tri-Sasakian space $X_7(\mathbf{t})$

The volume of hyperkahler quotient $\mathcal{M}$ and their equivariant deformation.
Suppose that a hyperkähler manifold $\mathcal{P}$ with three kahler forms $\boldsymbol{\omega}$ has a symmetry group G generated by tri-holomorphic vector fields V^a, and so $\mathcal{L}_{V^a}\boldsymbol{\omega} = d(i_{V^a}\boldsymbol{\omega}) + i_{V^a}d\boldsymbol{\omega} = 0$. As $d\boldsymbol{\omega} = 0$, there exists three moment map $\boldsymbol{\mu}^n$ such that

$$i_{V^a}\boldsymbol{\omega} \; = \; d\boldsymbol{\mu}^a \,. \tag{3.5}$$

The hyperkahler quotient space $M = \boldsymbol{\mu}^{-1}/G$ is again hyperkahler with induced three kahler forms.

Our objective in this subsection is to express the volume of the quotient $\mathcal{M}$ in terms of some integration over the ambient space $\mathcal{P}$, which we treat as the flat hyperkahler space $\mathcal{P} = \mathbb{H}^n \cong \mathbb{C}^{2n}$. In this work, we treat the quotient group to be a single abelian group, but the generalization to non-abelian groups is similar. Our work here is a straightforward application of the method in ref. [23].

We can pick any kahler form out of $\boldsymbol{\omega}$, say $\omega(x) \equiv \omega^3 = \omega_{\mu\nu}(x)dx^\mu \wedge dx^\nu/2$, to define the volume of a hyperkahler manifold $\mathcal{P}$,

$$vol(\mathcal{P}) = \int_{\mathcal{P}} e^\omega = \frac{1}{(2n)!} \int_{\mathcal{P}} \omega^{2n} \,, \tag{3.6}$$

where $\dim_C \mathcal{P} = 2n$. The normalization at this point is obscure, but later we will fix it to reproduce the flat volume of $\mathbb{H}^n \cong \mathbb{C}^{2n}$. Fixing normalization for the ambient space then unambiguously determine that of the quotient space.

We introduce mutually anti-commuting Grassmann variables ψ^μ which replace the 1-form variable dx^μ, and rewrite the kahler form as $\omega(x,\psi) = \omega_{\mu\nu}(x)\psi^\mu\psi^\nu/2$, which is a function of a bosonic coordinates x^μ and fermionic variables ψ^μ. Any differential form f in the space of differential forms, $\Omega^*(\mathcal{P})$, can be regard as a function $f(x,\psi)$. We can consider (x,ψ) as parameterizing a supermanifold $\mathcal{P}'$, in which usual tangent space is fermionic rather than bosonic, say $\psi^\mu \partial/\partial x^\mu$. With this notation, the integration of a top differential form f on $\mathcal{P}$ is written in a way that mimics supersymmetric functional integration,

$$\int_{\mathcal{P}} f = \int_{\mathcal{P}'} dx^1 dx^2 \cdots dx^{4n} d\psi^1 d\psi^2 \cdots d\psi^{4n} \, f(x,\psi) = \int_{\mathcal{P}'} [dx][d\psi] \, f(x,\psi) \,. \tag{3.7}$$

It can be checked by calculating super-Jacobian that the measure $[dx][d\psi]$ is invariant under coordinate reparametrization, and so is well-defined everywhere on $\mathcal{P}$. The volume formula in (3.6) is then written

$$vol(\mathcal{P}) = \int_{\mathcal{P}'} [dx][d\psi] \, e^{\omega} \,. \tag{3.8}$$

Note that $\int [dx][d\psi]$ automatically picks up the top dimensional form in the expansion of e^{ω} due to the properties of the Grassmann integration.

Now let us consider the volume of the hyperkahler quotient space $\mathcal{M}$ of the flat space $\mathcal{P}$ by a $\mathrm{U}(1)$ action, which is generated by the Killing vector $V = V^{\mu}(x)\partial/\partial x^{\mu}$ which preserves the triplet of kahler forms,

$$\mathcal{L}_V \boldsymbol{\omega} = 0 \ \longrightarrow \ i_V \boldsymbol{\omega} = d\boldsymbol{\mu} \tag{3.9}$$

The hyperkahler quotient space $\mathcal{M}$ is defined as $\mathcal{M} = \boldsymbol{\mu}^{-1}(0)/\mathrm{U}(1)$, which is again hyperkahler with kahler forms naturally inherited from $\mathcal{P}$.

To describe the quotient procedure more explicitly, we first note that $\mathcal{L}_V \boldsymbol{\mu} = d\boldsymbol{\mu}(V) = i_V \boldsymbol{\omega}(V) = \boldsymbol{\omega}(V, V) = 0$, which shows that the level surface $\boldsymbol{\mu}^{-1}(0)$ is invariant under $\mathrm{U}(1)$ flow. Then the Killing vector V^{μ} is parallel to $\boldsymbol{\mu}^{-1}(0)$. We can therefore introduce a local coordinate system (x^i, x^v) on (4n-3)-dim space $\boldsymbol{\mu}^{-1}(0)$ such that $V = \partial/\partial x^v$, and x^i $(i = 1, \ldots, 4n - 4)$ are constant along $\mathrm{U}(1)$ trajectories. As we further quotient along the direction of V to get $\mathcal{M}$, we naturally identify x^i as a coordinate system on $\mathcal{M}$. In the ambient space $\mathcal{P}$, $\boldsymbol{\mu}^{-1}(0)$ is codimension 3, so we locally introduce x^{ℓ} $(\ell = 1, 2, 3)$ around $\boldsymbol{\mu}^{-1}(0)$ as the coordinates along normal directions. In the following, we only consider points on $\boldsymbol{\mu}^{-1}(0)$ (or $x^{\ell} = 0$) unless stated otherwise. In components, the equation $i_V \boldsymbol{\omega} = d\boldsymbol{\mu}$ becomes

$$\boldsymbol{\omega}_{vi} dx^i + \boldsymbol{\omega}_{vn} dx^n = \frac{\partial \boldsymbol{\mu}}{\partial x^i} dx^i + \frac{\partial \boldsymbol{\mu}}{\partial x^{\ell}} dx^{\ell} = \partial_i \boldsymbol{\mu} \, dx^i + \partial_{\ell} \boldsymbol{\mu} \, dx^{\ell} \tag{3.10}$$

Because $\boldsymbol{\mu} = 0$ on $\boldsymbol{\mu}^{-1}(0)$, we have $\partial_i \boldsymbol{\mu} = 0$, and we get

$$\boldsymbol{\omega}_{vi} = 0 \,, \qquad \boldsymbol{\omega}_{v\ell} = \partial_{\ell} \boldsymbol{\mu} \tag{3.11}$$

From the closed-ness equation $d\boldsymbol{\omega} = 0$, we have $\partial_v \boldsymbol{\omega}_{ij} = \partial_i \boldsymbol{\omega}_{vj} - \partial_j \boldsymbol{\omega}_{vi}$, but $\boldsymbol{\omega}_{vi} = 0$ on every $\boldsymbol{\mu}^{-1}(0)$ and its tangent derivatives also vanish, so that $\partial_v \boldsymbol{\omega}_{ij} = 0$. This means that $\boldsymbol{\omega}_{\mathcal{M}} = \boldsymbol{\omega}_{ij} dx^i \wedge dx^j / 2$ does not vary along V, and so is well defined on $\mathcal{M}$, and so is identified as the induced triplet kahler forms.

As we have identified a coordinate system and triplet kahler forms on the quotient space $\mathcal{M}$, its symplectic volume will be

$$vol(\mathcal{M}) = \int_{\mathcal{M}'} [dx^i][d\psi^i] \, e^{\frac{1}{2}\omega_{ij}\psi^i\psi^j} \,, \tag{3.12}$$

where $\omega_{ij} = \omega_{ij}^3(x^i)$ on $\boldsymbol{\mu}^{-1}(0)$ and $\mathcal{M}'$ is the corresponding supermanifold for $\mathcal{M}$. Our aim is to rewrite (3.12) as an integration over the ambient space $\mathcal{P}$, which doesn't depend

manifestly on a particular coordinate system we have chosen in the above. Firstly, because ω_{ij} is independent of x^v, the integral can be extended to $\boldsymbol{\mu}^{-1}(0)$ as

$$\int_{\mathcal{M}'} [dx^i][d\psi^i]\, e^{\frac{1}{2}\omega_{ij}\psi^i\psi^j} = \frac{1}{vol(\mathrm{U}(1))} \int_{\boldsymbol{\mu}^{-1}(0)} [dx^i][dx^v][d\psi^i]\, e^{\frac{1}{2}\omega_{ij}\psi^i\psi^j}\,, \tag{3.13}$$

where $vol(\mathrm{U}(1)) = \int dx^v$ is the range of the coordinate x^v. To confine the integration on $\mathcal{P}$ onto $\boldsymbol{\mu}^{-1}(0) = \{x^n = 0\}$, we would need a δ-function factor $\prod_{a=1}^3 \delta(\mu^a(x))$, and to correctly reduce $\int [dx^i][dx^v][dx^n]$ into $\int [dx^i][dx^v]$, we have to add a Jacobian factor,

$$\int_{\boldsymbol{\mu}^{-1}(0)} [dx^i][dx^v] = \int_{\mathcal{P}} [dx^i][dx^v][dx^\ell] \prod_{a=1}^3 \delta(\mu^a(x)) \det\left(\partial_\ell \mu^a\right)$$

$$= \frac{1}{(2\pi)^3} \int_{\mathcal{P}} [dx^i][dx^v][dx^n][d\psi^\ell][d\phi_a][d\chi_a]\, e^{i\phi_a\mu^a + \chi_a(\partial_\ell\mu^a)\psi^\ell}\,, \tag{3.14}$$

where ϕ_a are bosonic and χ_a, ψ^ℓ are fermionic variables. From (3.11), we have $\partial_\ell\mu^3 = \omega_{v\ell}^3 = \omega_{v\ell}$, and calling $\chi_3 \equiv \psi^v$, we have $\chi_3(\partial_\ell\mu^3)\psi^n = \omega_{v\ell}\psi^v\psi^\ell = \omega_{v\mu}\psi^v\psi^\mu$ since $\omega_{vi} = 0$ on $\boldsymbol{\mu}^{-1}(0)$. Similarly $\chi_1\partial_\ell\mu^1\psi^\ell + \chi_2\partial_\ell\mu^2\psi^\ell = \chi_1\partial_\mu\mu^1\psi^\mu + \chi_2\partial_\mu\mu^2\psi^\mu \equiv \chi_1 d\mu^1 + \chi_2 d\mu^2$ since $\partial_v\mu^a = 0, \partial_i\mu^a = 0$ on $\boldsymbol{\mu}^{-1}(0)$. Therefore, the volume of $\mathcal{M}$ is written as

$$vol(\mathcal{M}) = \frac{1}{(2\pi)^3\, vol(\mathrm{U}(1))} \int_{\mathcal{P}''} \quad [dx^i][dx^v][dx^n][d\psi^i][d\psi^v][d\psi^n][d\phi_a][d\chi_{1,2}]$$

$$\times\; e^{\frac{1}{2}\omega_{ij}\psi^i\psi^j + \omega_{v\mu}\psi^v\psi^\mu + i\phi_a\mu^a + \chi_1 d\mu^1 + \chi_2 d\mu^2}\,. \tag{3.15}$$

where $\mathcal{P}''$ is the space a bit bigger than the supermanifold with additional coordinates $\phi_a, \chi_{1,2}$. The terms in the exponent involving ω almost comprise the Kähler form $\frac{1}{2}\omega_{\mu\nu}\psi^\mu\psi^\nu$ on $\mathcal{P}$, except $\omega_{in}\psi^i\psi^n + \frac{1}{2}\omega_{mn}\psi^m\psi^n$. However, the first term can be removed by shifting ψ^i, and also the second term by shifting χ_a. Therefore, we can replace the exponent by $\frac{1}{2}\omega_{\mu\nu}\psi^\mu\psi^\nu + i\phi_a\mu^a + \chi_1 d\mu^1 + \chi_2 d\mu^2$ without changing the result. Finally, the measure involves $[dx][d\psi]$ on $\mathcal{P}$, which is manifestly independent of a particular coordinate choice. Thus, we have

$$vol(\mathcal{M}) = \frac{1}{(2\pi)^3\, vol(\mathrm{U}(1))} \int_{\mathcal{P}''} [dx][d\psi][d\phi_a][d\chi_{1,2}]\; e^S\,, \tag{3.16}$$

where the 'action' is

$$S(x, \psi, \phi_a, \chi_1, \chi_2) = \omega + i\phi_a\mu^a + \chi_1 d\mu^1 + \chi_2 d\mu^2 \tag{3.17}$$

$$= \frac{1}{2}\omega_{\mu\nu}(x)\psi^\mu\psi^\nu + i\phi_a\mu^a(x) + \chi_1\partial_\mu\mu^1(x)\psi^\mu + \chi_2\partial_\mu\mu^2(x)\psi^\mu\,.$$

The above integration looks like a path integral of a (0+0)-dimensional supersymmetric system. Indeed, the action S has the following fermionic symmetry,

$$\begin{aligned}
Q\, x^\mu &= \psi^\mu\,, & Q\, \psi^\mu &= -i\phi_3 V^\mu(x)\,, \\
Q\, \phi_a &= 0\,, & Q\, \chi_1 &= -i\phi_1\,, \\
Q\, \chi_2 &= -i\phi_2\,,
\end{aligned} \tag{3.18}$$

which can be easily verified using $d\omega = 0$ and $i_V \boldsymbol{\omega} = d\boldsymbol{\mu}$. Acting Q twice, we have $Q^2 x^\mu = -i\phi_3 V^\mu$ and $Q^2 \psi^\mu = -i\phi_3(\partial_\nu V^\mu)\psi^\nu$, while $Q^2(\phi, \chi) = 0$. On the space of functions on $\Omega^*(\mathcal{P})$, that is, on the space of differential forms on $\mathcal{P}$, this is nothing but $Q^2 = -i\phi_3 \mathcal{L}_V$. Therefore, if we restrict to the space of U(1)-invariant differential forms, Q is nilpotent. In fact, the right observables well-defined on $\mathcal{M}$ are indeed U(1)-invariant differential forms on $\mathcal{P}$, and the correlation functions of them depend only on the Q-cohomology. These correlation functions are nothing but the integrals on $\mathcal{M}$ performed in the ambient space $\mathcal{P}$.

Suppose that there is a U(1)$_R$-action on $\mathcal{P}$ generated by $R^\mu(x)$ such that $\mathcal{L}_R\omega = \mathcal{L}_R\omega^3 = 0$, $\mathcal{L}_R(\omega^1 - i\omega^2) = 2i(\omega^1 - i\omega^2)$, which imply that $\mathcal{L}_R\mu^3 = R^\alpha\partial_\alpha\mu^3 = 0$, and $\mathcal{L}_R(\mu^1 - i\mu^2) = R^\alpha\partial_\alpha(\mu^1 - i\mu^2) = 2i(\mu^1 - i\mu^2)$. The former implies that there is a function $H(x)$ with $i_R\omega = dH$. We also assume that V commutes with R, that is, $[V, R] = 0$. Then we naturally assign the R-action to ϕ_a and $\chi_{1,2}$ such that the integrand S respects this symmetry;

$$R \cdot (\phi_1 - i\phi_2) = 2i(\phi_1 - i\phi_2), \quad R \cdot (\chi_1 - i\chi_2) = 2i(\chi_1 - i\chi_2), \tag{3.19}$$

and $R \cdot \phi_3 = 0$. This allows us to deform the supersymmetry Q to Q_ϵ in the following way,

$$
\begin{aligned}
Q_\epsilon x^\mu &= \psi^\mu, & Q_\epsilon \psi^\mu &= -i\phi_3 V^\mu(x) + \epsilon R^\mu(x), & Q_\epsilon \phi_3 &= 0, \\
Q_\epsilon \phi_1 &= 2i\epsilon\chi_2, & Q_\epsilon \phi_2 &= -2i\epsilon\chi_1, & Q_\epsilon \chi_1 &= -i\phi_1, \\
& & Q_\epsilon \chi_2 &= -i\phi_2,
\end{aligned}
\tag{3.20}
$$

with $Q_\epsilon^2 = -i\phi_3\mathcal{L}_V + \epsilon R$, where R acts as $\mathcal{L}_R$ on differential forms, and ϵ is a constant. Thus, $Q_\epsilon^2 = 0$ still holds on the space of both U(1) and U(1)$_R$ invariant functions. It is straightforward to check that S is both U(1) and U(1)$_R$ invariant, and

$$Q_\epsilon S = \epsilon \omega_{\mu\nu} R^\mu(x)\psi^\nu = \epsilon\, i_R\omega = \epsilon\, dH = Q_\epsilon(\epsilon H(x)). \tag{3.21}$$

Hence, $S_\epsilon \equiv S - \epsilon H$ is Q_ϵ-invariant. For non-compact hyperkahler manifolds, the term $-\epsilon H$ often provides a regularization for volume [23]. In addition, the regularized volume integration would be left unchanged if we add a bosonic Q_ϵ-exact term $Q_\epsilon\mathcal{O}$ to the deformed action S_ϵ, where fermionic $\mathcal{O}$ should be U(1) and U(1)$_R$-invariant, so that $Q_\epsilon^2\mathcal{O} = 0$. We then consider the regularized volume of $\mathcal{M}$,

$$vol_\epsilon(\mathcal{M}) = \frac{1}{(2\pi)^3\, vol(\mathrm{U}(1))} \int_{\mathcal{P}''} [dx][d\psi][d\phi_a][d\chi_{1,2}]\, e^{S - \epsilon H + Q_\epsilon\mathcal{O}'}. \tag{3.22}$$

One useful $Q_\epsilon\mathcal{O}'$ is

$$Q_\epsilon\mathcal{O}' = Q_\epsilon\left(-it(\chi_1\phi_1 + \chi_2\phi_2)\right) = -t(\phi_1\phi_1 + \phi_2\phi_2) - 4\epsilon t(\chi_1\chi_2), \tag{3.23}$$

which will dominate the $\phi_{1,2}$ and $\chi_{1,2}$ terms in the action when we take $t \to \infty$ limit, which allows a simple integration over $\phi_{1,2}, \chi_{1,2}$. The remaining integration over x^μ, ψ^μ and ϕ_3 will then be simple Gaussian in our case and can be performed easily.

Calculations for $X_7(\mathbf{t})$. We now apply the previous formalism to an explicit problem of our space $X_7(\mathbf{t}) = S(t_1, t_2, t_3)$. We started with a hyperkähler quotient of the flat 12-dim hyperkahler space $\mathcal{P} = \mathbb{H}^3 = (q_1, q_2, q_3)$. We obtained the 8-dim hyperkähler space $\mathcal{M}$ by the hyperkahler quotient of $\mathcal{P}$ by a U(1) action $q_a \to q_a e^{i\sigma_3 t_a \xi}$, $a = 1, 2, 3$. Here we use the representation of the quaternions q_a by the complex coordinates u_a, v_a as in eq. (2.4), which in turn be represented by the real coordiantes

$$q_a = \begin{pmatrix} u_a & v_a \\ -\bar{v}_a & \bar{u}_a \end{pmatrix} = \begin{pmatrix} x_a + iy_a & \tilde{x}_a + i\tilde{y}_a \\ -\tilde{x}_a + i\tilde{y}_a & x_a - iy_a \end{pmatrix} . \tag{3.24}$$

The triplet hyperkahler forms (2.3) become

$$\omega^3 = -(dx_a \wedge dy_a + d\tilde{x}_a \wedge d\tilde{y}_a) ,$$
$$\omega^1 - i\omega^2 = i(dx_a \wedge d\tilde{x}_a - dy_a \wedge d\tilde{y}_a) - (dx_a \wedge d\tilde{y}_a + dy_a \wedge d\tilde{x}_a) . \tag{3.25}$$

With these canonical coordinates, the volume $\int [dx][d\psi] e^\omega$ of the ambient space $\mathcal{P} = \mathbb{H}^3$ is simply the flat volume $\int [dx_a][dy_a][d\tilde{x}_a][d\tilde{y}_a]$. This fixes the normalization of the ambient space metric to be $ds^2_{\mathbb{H}^3} = dx_a dx_a + dy_a dy_a + d\tilde{x}_a d\tilde{x}_a + d\tilde{y}_a d\tilde{y}_a$. In components, the above U(1) action is

$$u_a \to e^{it_a\xi} u_a , \quad v_a \to e^{-it_a\xi} v_a , \tag{3.26}$$

so that the generating vector field V is

$$V = \frac{\partial}{\partial \xi} = t_a \left(x_a \frac{\partial}{\partial y_a} - y_a \frac{\partial}{\partial x_a} \right) - t_a \left(\tilde{x}_a \frac{\partial}{\partial \tilde{y}_a} - \tilde{y}_a \frac{\partial}{\partial \tilde{x}_a} \right) . \tag{3.27}$$

From the definition of $i_V \boldsymbol{\omega} = d\boldsymbol{\mu}$, we have

$$\mu^3 = \frac{1}{2} t_a \left(|u_a|^2 - |v_a|^2 \right) , \quad \mu^1 - i\mu^2 = -t_a u_a v_a . \tag{3.28}$$

In addition, there is a diagonal $U(1)_R$ action of $SU(2)_R$ with the fore-mentioned properties,

$$u_a \to e^{i\epsilon} u_a , \quad v_a \to e^{i\epsilon} v_a , \tag{3.29}$$

which gives us R as

$$R = \left(x_a \frac{\partial}{\partial y_a} - y_a \frac{\partial}{\partial x_a} + \tilde{x}_a \frac{\partial}{\partial \tilde{y}_a} - \tilde{y}_a \frac{\partial}{\partial \tilde{x}_a} \right) , \tag{3.30}$$

and from $i_R \omega = dH$, we have $H = \frac{1}{2}(|u_a|^2 + |v_a|^2) = \frac{1}{2} r^2$ with r being the standard radial distance in the above flat metric.

The volume of U(1) is the coordinate length of ξ, that is, the least number ξ such that $t_a \xi \in 2\pi\mathbb{Z}$ for all $a = 1, 2, 3$. It is easily seen to be

$$vol(U(1)) = (2\pi) l.c.m \left(\frac{1}{t_a} \right) = \frac{2\pi}{t_1 t_2 t_3} l.c.m(t_1 t_2, t_2 t_3, t_3 t_1) , \tag{3.31}$$

where $l.c.m$ stands for least common multiple.

Let us integrate over $\phi_{1,2}$ and $\chi_{1,2}$ in the regularized volume integration (3.22) in the large t limit, $\mathcal{M}_8$. The integration is dominated by $Q_\epsilon \mathcal{O}$ to give

$$\int d\phi_1 d\phi_2 d\chi_1 d\chi_2 \, e^{-t(\phi_1\phi_1+\phi_2\phi_2)-4\epsilon t(\chi_1\chi_2)} = \frac{\pi}{t} \cdot 4\epsilon t = 4\pi\epsilon \,. \tag{3.32}$$

The remaining integration is

$$vol_\epsilon(\mathcal{M}_8(\mathbf{t})) = \frac{4\pi\epsilon}{(2\pi)^3 \, vol(\mathrm{U}(1))} \int_{\mathbb{H}^3} [dx][d\psi]d\phi_3 \, e^{\frac{1}{2}\omega_{\mu\nu}\psi^\mu\psi^\nu+i\phi_3\mu^3(x)-\epsilon H(x)} \,. \tag{3.33}$$

Note that ω in (3.25) is simply constant in the flat coordinate $(x_a, y_a, \tilde{x}_a, \tilde{y}_a)$ of $\mathcal{P}$, and the $[d\psi]$ integration readily calculated to be

$$\int [d\psi] \, e^{\frac{1}{2}\omega_{\mu\nu}\psi^\mu\psi^\nu} = 1 \,. \tag{3.34}$$

Also, μ^3 and H are both Gaussian functions on x^μ and a simple calculation gives

$$\int_{\mathbb{H}^3} [dx] \, e^{i\phi_3\mu^3-\epsilon H} = \int [dx] \, e^{\frac{i}{2}\phi_3 t_a\left((x_a)^2+(y_a)^2-(\tilde{x}_a)^2-(\tilde{y}_a)^2\right)-\frac{\epsilon}{2}\left((x_a)^2+(y_a)^2+(\tilde{x}_a)^2+(\tilde{y}_a)^2\right)}$$

$$= (2\pi)^6 \prod_{a=1}^3 \frac{1}{(\epsilon-it_a\phi_3)(\epsilon+it_a\phi_3)} \,, \tag{3.35}$$

so that

$$vol_\epsilon(\mathcal{M}_8(\mathbf{t})) = \frac{4\pi\epsilon(2\pi)^3}{vol(\mathrm{U}(1))} \int d\phi_3 \prod_{a=1}^3 \frac{1}{(\epsilon-it_a\phi_3)(\epsilon+it_a\phi_3)} \,. \tag{3.36}$$

The ϕ_3 integration has poles at $\phi_3 = \pm\frac{i\epsilon}{t_a}$, and by closing the contour to the upper half plane, we pick up poles at $\phi_3 = \frac{i\epsilon}{t_a}$, $a = 1, 2, 3$. The result is

$$\int d\phi_3 \prod_{a=1}^3 \frac{1}{(\epsilon-it_a\phi_3)(\epsilon+it_a\phi_3)} = \frac{\pi}{\epsilon^5} \frac{t_1t_2+t_2t_3+t_3t_1}{(t_1+t_2)(t_2+t_3)(t_3+t_1)} \,. \tag{3.37}$$

In summary, we have the regularized volume of the 8-dim quotient space

$$vol_\epsilon(\mathcal{M}_8(\mathbf{t})) = \frac{16\pi^4}{\epsilon^4} \frac{t_1t_2t_3(t_1t_2+t_2t_3+t_3t_1)}{l.c.m(t_1t_2, t_2t_3, t_3t_1)(t_1+t_2)(t_2+t_3)(t_3+t_1)} \,. \tag{3.38}$$

While we started with three distinct natural numbers t_a, the above formula is well defined for any postive real number t_a and so can be regarded valid even when some of t_a coincide. (Note the above procedure can be easily generalized to the regularized volume for the hyperkahler quotient space $\mathcal{M}_{4n}$ obtained from the $\mathbb{H}^{n+1}$ by a single $\mathrm{U}(1)$).

As our objective is to calculate the volumes of 7-dim tri-Sasakian section of 8-dimensional hyperkahler cones, this Hamiltonian regularization happens to be exactly what we would need to extract the volumes of tri-Sasakian section. This is because, in our cases at hand, H turns out to be $H = \frac{1}{2}r^2$, when the metric is written as $ds_8^2 = dr^2 + r^2 ds_{X_7}^2$, and the regularized volume is

$$vol_\epsilon(\mathcal{M}_8) = vol(X_7) \int_0^\infty r^7 e^{-\frac{1}{2}\epsilon r^2} = \frac{48}{\epsilon^4} vol(X_7) \quad . \tag{3.39}$$

Finding the regularized volume would give us the volume of the tri-Sasakian section with normalization $R_{ij} = 6g_{ij}$.

By using (3.39), we obtain the formula

$$\frac{vol(X_7(\mathbf{t}))}{vol(S^7)} = \frac{t_1 t_2 t_3 (t_1 t_2 + t_2 t_3 + t_3 t_1)}{l.c.m(t_1 t_2, t_2 t_3, t_3 t_1)(t_1 + t_2)(t_2 + t_3)(t_3 + t_1)}. \tag{3.40}$$

The volume of the unit 7 sphere is $vol(S^7) = \pi^4/3$. The ratio of two volumes is a rational number. As a check, the well-known space $N(1,1)$ corresponds to $t_1 = t_2 = t_3 = 1$ for which we reproduce the known answer $vol(N(1,1)) = \frac{\pi^4}{8}$. We can show that the right hand side is less than $3/8$. The reason is that for three natural numbers t_1, t_2, t_3 the following inequalties hold,

$$t_1 t_2 + t_2 t_3 + t_3 t_1 \leq 3 \, l.c.m(t_1 t_2, t_2 t_3, t_3 t_1),$$
$$t_1 t_2 t_3 \leq \frac{1}{8}(t_1 + t_2)(t_2 + t_3)(t_3 + t_1), \tag{3.41}$$

where the last inequality comes from $2\sqrt{t_1 t_2} \leq t_1 + t_2$ and so on. The equality holds only when $t_1 = t_2 = t_3$.

One could study the tri-Sasakian space $\mathcal{M}_{4n-1}$ obtained from the hyperkahler quotient of $\mathbb{H}^{n+1}$ by a similar $U(1)_{\mathbf{t}}$ group. The volume can be calculated by the above method and is equal to

$$\frac{vol(M_{4n-1})}{vol(S_{4n-1})} = \frac{1}{l.c.m.(1/t_1, \ldots, 1/t_{n+1})} \sum_a \frac{t_a^{2n-1}}{\prod_{b \neq a}(t_a^2 - t_b^2)}, \tag{3.42}$$

where the volume of the unit sphere $vol(S_{2n-1})$ is $(2n-1)! \, \pi^n/2$.

3.2 The volumes of supersymmetric 5-cycle Σ_5

The cone of the tri-Sasakian spaces $X_7 = S(t_1, t_2, t_3)$ are 8-dimensional hyperkähler spaces, which are constructed through hyperkahler quotient. A supersymmetric 5-cycle, Σ_5 in $S(t_1, t_2, t_3)$ is characterized by its cone Γ, which is a 6-dimensional subspace of the hyperkähler cone $\mathcal{M}(\mathbf{t})$, defined by a single homogeneous holomorphic constraint. In fact, the two 5-cycles in $N(1,1) = S(1,1,1)$ that were identified in ref. [17] correspond to $u_3 = 0$ and $v_3 = 0$ respectively. Of course, there are others such as $u_1 = 0$ et cetera, which are related to each other by continuous SU(3) isometry. They necessarily belong to the same homology. For generic $S(t_1, t_2, t_3)$, the constraint $u_a = 0$ or $v_a = 0$ for some a again defines a supersymmetric 5-cycle, but the flavor isometry is now reduced to $U(1)^2$ and the cycles with different a's are separated by potential walls. The remaining $SU(2)_R$ symmetry still relates $u_a = 0$ and $v_a = 0$ for the same a.

In this section, we calculate the volumes of supersymmetric 5-cycles using the formalism of the previous section. Without loss of generality, we specify to a supersymmetric 5-cycle Σ_5 obtained from the constraint, say with $u_3 = 0$, whose 6-dim cone is Γ. Before considering Γ in our hyperkahler quotient space, let us consider the 10-cycle $\tilde{\Gamma}$ defined in the ambient space $\mathbb{H}^3$ by the same constraint $u_3 = 0$. Its volume, though infinite, is expressed formally as

$$\frac{1}{5!} \int_{u_3 = 0} \omega^5 = \frac{1}{5!} \int_{\mathbb{H}^3} \Phi_{\tilde{\Gamma}} \wedge \omega^5, \tag{3.43}$$

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where ω is the Kahler form, and $\Phi_{\tilde{\Gamma}}$ is the 2-form Thom class dual to $\tilde{\Gamma}$. In the real coordinate system introduced in the last section, $u_3 = x_3 + iy_3$ and $\Phi_{\tilde{\Gamma}} = \delta(x_3)\delta(y_3)dx_3 \wedge dy_3$. It is easily verified that $d\Phi_{\tilde{\Gamma}} = 0$. In the formalism of the previous section where differential forms are functions on $\mathcal{P}$, we can rewrite the above as an expectation value,

$$vol(\tilde{\Gamma}) = \langle \Phi_{\tilde{\Gamma}} \rangle = \int_{\mathbb{H}^{3\prime}} [dx][d\psi] \, \Phi_{\tilde{\Gamma}} \, e^{\omega} \,, \tag{3.44}$$

where $\Phi_{\tilde{\Gamma}} = \delta(x_3)\delta(y_3)\psi^{x_3}\psi^{y_3}$ and $\omega = \omega_{\mu\nu}\psi^{\mu}\psi^{\nu}/2$. The 'action' $S = \omega$ is invariant under a fermionic symmetry $Qx^{\mu} = \psi^{\mu}$, $Q\psi^{\mu} = 0$ due to $d\omega = 0$ (Q is in fact the de Rham d-operator on differential forms). Because $Q\Phi_{\tilde{\Gamma}} = 0$ ($d\Phi_{\tilde{\Gamma}} = 0$), $\Phi_{\tilde{\Gamma}}$ is a good observable of the above path integral.

For our quotient space $\mathcal{M}_8(\mathbf{t})$, we have represented the regularized volume as a path integral

$$vol_{\epsilon}(\mathcal{M}_8(\mathbf{t})) = \langle \, 1 \, \rangle_{\epsilon} = \frac{1}{(2\pi)^3 \, vol(\mathrm{U}(1))} \int [dx][d\psi][d\phi_a][d\chi_{1,2}] \, e^{S - \epsilon H + Q_{\epsilon}\mathcal{O}'} \,. \tag{3.45}$$

It also has a fermionic symmetry Q_{ϵ} as discussed before. Actually, there are two separate components in S_{ϵ} that are Q_{ϵ}-invariant; one is $\frac{1}{2}\omega + i\phi_3\mu^3 - \epsilon H$, the other being $i\phi_1\mu^1 + i\phi_2\mu^2 + \chi_1 d\mu^1 + \chi_2 d\mu^2$. In the spirit of equivariant cohomology, a good observable $\mathcal{O}$ satisfying $Q_{\epsilon}\mathcal{O}$ represents a well-defined geometric data on the quotient space. In this respect, the first piece may be considered as the Kahler form of the quotient space, and our path integral naturally calculates the regularized volume of the quotient space.

Taking this analogy further, we expect there should exist the right observable Φ_{Γ} whose expectation value calculates the regularized volume of Γ defined by $u_3 = 0$ in the quotient space. In the ambient space $\mathbb{H}^3$, it is $\Phi_{\tilde{\Gamma}}$ and we naturally expect that for quotient space, it would be some modification of $\Phi_{\tilde{\Gamma}}$ such that it is Q_{ϵ}-invariant. In our case at hand, it is readily shown that $\Phi_{\tilde{\Gamma}}$ itself is Q_{ϵ}-invariant because of δ-function factors. Therefore, we take $\Phi_{\Gamma} = \Phi_{\tilde{\Gamma}}$, and the regularized volume of Γ will be

$$vol_{\epsilon}(\Gamma) = \langle \Phi_{\Gamma} \rangle_{\epsilon} = \frac{1}{(2\pi)^3 \, vol(\mathrm{U}(1))} \int [dx][d\psi][d\phi_a][d\chi_{1,2}] \, \Phi_{\Gamma} \, e^{S - \epsilon H + Q_{\epsilon}\mathcal{O}'} \tag{3.46}$$

From $vol_{\epsilon}(\Gamma)$, it is straightforward to extract the volume of 5-dimensional cycle Σ_5 that we are heading to. Writing the metric on Γ as $ds_{\Gamma}^2 = dr^2 + r^2 ds_{\Sigma_5}^2$,

$$vol_{\epsilon}(\Gamma) = vol(\Sigma_5) \int_0^{\infty} dr \, r^5 \, e^{-\frac{1}{2}\epsilon r^2} = vol(\Sigma_5) \cdot \frac{8}{\epsilon^3} \,. \tag{3.47}$$

The calculation of (3.46) is almost same as the one in the previous section. Introducing large Q_{ϵ}-exact mass term (3.23) for $\phi^{1,2}$ and $\chi^{1,2}$, and integrating them out results in

$$vol_{\epsilon}(\Gamma) = \frac{4\pi\epsilon}{(2\pi)^3 vol(\mathrm{U}(1))} \int_{\mathbb{H}^3} [dx][d\psi]d\phi_3 \, \delta(x_3)\delta(y_3)\psi^{x_3}\psi^{y_3} \, e^{\omega + i\phi^3\mu^3(x) - \epsilon H(x)} \,. \tag{3.48}$$

By using simple integrations

$$\int [d\psi] \, \psi^{x_3}\psi^{y_3} \, e^{\frac{1}{2}\omega_{\mu\nu}\psi^{\mu}\psi^{\nu}} = 1 \,, \tag{3.49}$$

$$\int_{\mathbb{H}^3} [dx] \, \delta(x_3)\delta(y_3) \, e^{i\phi^3\mu^3 - \epsilon H} = (2\pi)^5 \frac{1}{(\epsilon - it_1\phi^3)(\epsilon - it_2\phi^3)\prod_{a=1}^3 (\epsilon + it_a\phi^3)} \,,$$

we are left with

$$vol_\epsilon(\Gamma) = \frac{4\pi\epsilon(2\pi)^2}{vol(\mathrm{U}(1))} \int d\phi^3 \, \frac{1}{(\epsilon - it_1\phi^3)(\epsilon - it_2\phi^3)\prod_{a=1}^3(\epsilon + it_a\phi^3)} \,. \tag{3.50}$$

The integrand has poles at $-\frac{i\epsilon}{t_1}, -\frac{i\epsilon}{t_2}$ and $+\frac{i\epsilon}{t_a}, a = 1, 2, 3$. Because it is convergent, we can close the contour in any way, and the result is

$$\int d\phi^3 \, \frac{1}{(\epsilon - it_1\phi^3)(\epsilon - it_2\phi^3)\prod_{a=1}^3(\epsilon + it_a\phi^3)} = \frac{\pi}{\epsilon^4} \cdot \frac{t_1 t_2 + t_2 t_3 + t_3 t_1}{(t_1 + t_2)(t_2 + t_3)(t_3 + t_1)}, \tag{3.51}$$

so that

$$vol_\epsilon(\Gamma) = \frac{16\pi^4}{vol(\mathrm{U}(1))} \frac{t_1 t_2 + t_2 t_3 + t_3 t_1}{(t_1 + t_2)(t_2 + t_3)(t_3 + t_1)} \cdot \frac{1}{\epsilon^3} = vol(\Sigma_5) \cdot \frac{8}{\epsilon^3} \,, \tag{3.52}$$

and we finally obtain

$$\frac{vol(\Sigma_5)}{vol(S^5)} = \frac{t_1 t_2 t_3 (t_1 t_2 + t_2 t_3 + t_3 t_1)}{l.c.m(t_1 t_2, t_2 t_3, t_3 t_1)(t_1 + t_2)(t_2 + t_3)(t_3 + t_1)} \,, \tag{3.53}$$

where the volume of unit five sphere is $vol(S^5) = \pi^3$. The above expression is identical to $vol(X_7)/vol(S^7)$. For $t_1 = t_2 = t_3 = 1$, it is $\frac{3\pi^3}{8}$ which agrees with the known value of supersymmetric 5-cycles of $N(1,1)$ in ref. [17].

Interestingly, the 5-cycles $u_a = 0$ or $v_a = 0$ have the same volume independent of a. The more striking fact is that the volume ratio between the supersymmetric 5-cycle and the total tri-Sasakian 7-manifold is independent of t_a and takes a universal value

$$\frac{vol(\Sigma_5)}{vol(X_7(\mathbf{t}))} = \frac{vol(S^5)}{vol(S^7)} = \frac{3}{\pi} \,. \tag{3.54}$$

This will turn out to be consistent with the SCFT expectation.

4. Dual superconformal field theory

The bosonic Lagrangian for 11-dim supergravity is

$$2\kappa^2 \mathcal{L} = \sqrt{-G}\mathcal{R} - \frac{1}{2}F_4 \wedge *F_4 - \frac{1}{6}C_3 \wedge F_4 \wedge F_4 \,, \tag{4.1}$$

in the convention of ref. [17]. The 11-dim Planck length l_{11} would be given by $2\kappa^2 = (2\pi)^8 l_{11}^9$. In this convention, the $M2$ charge and $M5$ charges are given by the flux of F_4 as

$$N_2 = \frac{1}{(2\pi l_{11})^6} \int_{C_7} *F_4 \,, \quad N_5 = \frac{1}{(2\pi l_{11})^3} \int_{C_4} F_4 \,, \tag{4.2}$$

for some 7-cycle C_7 and 4-cycle C_4 surrounding branes. Our supersymmetric solution of the supergravity takes the form

$$ds_{11}^2 = R_{X_7}^2 \left(\frac{1}{4}ds_{AdS_4}^2 + ds_{X_7}^2\right) \,, \quad F_4 = \frac{3}{8}R_{X_7}^3 \, vol_{AdS_4} \,, \tag{4.3}$$

with vol_{AdS_4} being the volume form of the AdS_4 space with metric $ds^2_{AdS_4}$. Here we assume that the normalization of $ds^2_{X_7}$ is such that $R_{\mu\nu}(X_7) = 6g_{\mu\nu}(X_7)$. The radius of R_{X_7} is given by the quantization condition of M2 brane charge N_2,

$$6R^6_{X_7}vol(X_7) = (2\pi l_{11})^6 N_2\,, \tag{4.4}$$

where $vol(X_7)$ is the volume of X_7, which we calculate this volume in the section ahead.

The dual field theory on $M2$ branes on the singular point is a $\mathcal{N} = 3$ supersymmetric conformal field theory. One naively thinks that it is the infrared limit of a gauge theory with the product gauge group

$$SU(N)_1 \times SU(N)_2\,, \tag{4.5}$$

where N is the M2 charge N_2 of eq. (4.2). In terms of $\mathcal{N} = 2$ language ($\mathcal{N} = 1$ in 4-dim), there are 6 chiral fields u_a, v_a, $a = 1, 2, 3$. The best way to consider the $SU(2)_R$ symmetry is to group these fields into $U_a^\beta = (u_a, -\bar{v}_a)$ and $V_{a\beta} = (v_a, \bar{u}_a)$, where $a = 1, 2, 3$ and $\beta = 1, 2$. The chiral field U_a^β belongs to the symmetrized product representation $Sym^{t_a}(N)$ of the fundamental representation of the first $SU(N)_1$ and the symmetrized product representation $Sym^{t_a}(\bar{N})$ of the second $SU(N)_2$. The chiral field $V_{a\beta}$ transforms opposite to the chiral field U_a^β. Under the additional $U(1) \times U(1)$ global symmetries, the charges are shown in the following table.

	$SU(2)_R$	$SU(N)_1$	$SU(N)_2$	$U(1)$	$U(1)$
U_a^β	**2**	$Sym^{t_a}(N)$	$Sym^{t_a}(\bar{N})$	$(t_1, -t_2, 0)$	$(0, t_2, -t_3)$
$V_{a\beta}$	**2**	$Sym^{t_a}(\bar{N})$	$Sym^{t_a}(N)$	$(-t_1, t_2, 0)$	$(0, -t_2, t_3)$

As the R-symmetry is nonabelain $SU(2)$ group and these chiral fields belongs to the fundamental representation of this R-symmetry, these chiral fields would have a chiral dimension $1/2$. The chiral operators of dimension k would be, for example, traces of products of k U's and k V's with totally symmetric in $SU(2)_R$ indices. There are still many kinds of such operators with different combinations of flavor indices $a = 1, 2, 3$. In the case of $N(1, 1) = S(1, 1, 1)$, KK analysis of the geometry dictates only operators with totally symmetric and traceless in flavor indices [14], and we expect a similar kind of reduction in the spectrum for generic $S(t_1, t_2, t_3)$. In SCFT, we have to assume that only these operators survive in the IR fixed point. As was pointed out in ref. [15, 16], this contrasts to the case of $AdS_5 \times T^{1,1}$ where a superpotential in the dual field theory selects right chiral primary operators that match with gravity analysis [3].

We can try superpotential approach as much as in [16], though it would not be sufficient to determine chiral primary operators. We introduce the complex scalar fields Φ_1, Φ_2 in the adjoint representation of $SU(N)_1 \times SU(N)_2$, respectively. They belong to the vector multiplet and in component Φ_1^i, Φ_2^i with $i = 1, 2, \ldots, N^2 - 1$. We propose the superpotential to be

$$W = \sum_{a=1}^3 \left(g_1 \Phi_1^i \mathrm{Tr}_a T_a^i U_{a\beta} V^{a\beta} + g_2 \Phi_2^i \mathrm{Tr}_a T_a^i V_{a\beta} U_a^\beta \right) + k_1 \Phi_1^i \Phi_1^i + k_2 \Phi_2^i \Phi_2^i\,, \tag{4.6}$$

where $g_1 = g_2 = g$ are the gauge coupling constants, T_a^i are matrix representation in $Sym^{t_a}(N)$ and Tr_a is the trace operation in this representation. The Chern-Simons coefficients k_1 and k_2 may satisfy $k_1 = -k_2$ as in the case of $S(1,1,1) = N(1,1)$ [16]. Integrating out Φ's would produce a superpotential for U's and V's in IR.

The theory has a single U(1) baryon symmetry since the second and fifth betti numbers are $b_2(S(t_1, t_2, t_3)) = b_5(S(t_1, t_2, t_3)) = 1$. There are baryonic operators such as $\det U_a$ or $\det V_a$ where det represents the product of N U_a's or N V_a's totally anti-symmetrized in gauge indices for both $\mathrm{SU}(N)_1$ and $\mathrm{SU}(N)_2$. Then $\mathrm{SU}(2)_R$ indices are totally symmetric. The conformal dimension would then be $\Delta = \frac{N}{2}$ independent of a or $\mathbf{t}$. From the geometry, these operators correspond to M5 branes wrapping supersymmetric 5-cycles with the dimension (1.2) and the result (3.54) gives $\Delta = \frac{N}{2}$ which agrees with the gauge theory expectation. We take this as a non-trivial evidence for our proposal of superconformal field theory.

One should caution that the detail characteristics of the corresponding $\mathcal{N} = 3$ SCFT is very obscure. One naively can imagine that this theory is the low energy conformal limit of the Yang-Mills-Chern-Simons theory of the gauge group $\mathrm{SU}(N)_1 \times \mathrm{SU}(N)_2$ and the matter chiral fields u_a, v_a. In the low energy theory massive vector multiplet decouples, and so only the Chern-Simons kinetic term survives, with matter fields interact each other by the gauge coupling and self-coupling. The corresponding t'Hooft coupling is N/κ with κ being the Chern-Simons theory coefficient. From our geometric point of view of $AdS_4 \times X_7$, the Chern-Simons coefficient is not obvious at all. It would be interesting to find out further about this discrepancy.

Acknowledgments

A. A generalization of Caloron moduli space

We start this appendix by considering a generalization of the moduli space of distinct multi BPS magnetic monopole solutions [26]. Instead of considering the interaction between two BPS dyonic monopoles whose interactions are fixed by the Dynkin diagram for a give Lie algebra, we introduce a somewhat more general interaction between them. Thus the generalized Lagrangian between multi-monopoles would be

$$L = \frac{1}{2} M_{ij} \left(\dot{\mathbf{x}}_i \cdot \dot{\mathbf{x}}_j - q_i q_j \right) + q_i \mathbf{W}_{ij} \cdot \dot{\mathbf{x}}_j + q_i \dot{\xi}_i , \tag{A.1}$$

where

$$M_{ii} = m_i + \sum_{k \neq i} \frac{\lambda_{ik}}{|\mathbf{x}_i - \mathbf{x}_k|}, \quad M_{ij} = -\frac{\lambda_{ij}}{|\mathbf{x}_i - \mathbf{x}_j|} \text{ if } i \neq j , \tag{A.2}$$

with non-negative mass parameters $m_i \geq 0$ and

$$\mathbf{W}_{ii} = \sum_{k \neq i} \lambda_{ik} \mathbf{w}(\mathbf{x}_i - \mathbf{x}_k), \quad \mathbf{W}_{ij} = -\lambda_{ij} \mathbf{w}(\mathbf{x}_i - \mathbf{x}_j) \text{ if } i \neq j, \tag{A.3}$$

with $\mathbf{w}$ being the value at $\mathbf{x}_i$ of the Dirac potential due to the j-th monopole so that

$$\nabla \times \mathbf{w}(\mathbf{x}) = \frac{\mathbf{x}}{|\mathbf{x}|^3}. \tag{A.4}$$

The range of each phase is given by

$$0 \leq \xi_i < 4\pi t_i, \tag{A.5}$$

which implies that the quantization of charge is satisfied with

$$q_i = \frac{n_i}{2t_i} \tag{A.6}$$

with integer n_i. After integration over q_i, we obtain the Lagrangian

$$L = \frac{1}{2} M_{ij} \mathbf{v}_i \cdot \mathbf{v}_j + \frac{1}{2} M_{ij}^{-1} (\dot{\xi}_i + \mathbf{W}_{ik} \cdot \mathbf{v}_k)(\dot{\xi}_j + \mathbf{W}_{jl} \cdot \mathbf{v}_l). \tag{A.7}$$

The geometry is not necessarily regular when two point particles come together. Now we want the metric to be regular whenever any of two particles interacting each other come together. This requires more detail analysis of two bodies. With the center of mass coordinate for any two body, say $i = 1, 2$,

$$\mathbf{R} = \frac{m_1 \mathbf{x}_1 + m_2 \mathbf{x}_2}{m_1 + m_2}, \quad \mathbf{r} = \mathbf{x}_1 - \mathbf{x}_2. \tag{A.8}$$

The total charge and relative charge are defined as a linear combination

$$q_t = \frac{m_1 q_1 + m_2 q_2}{m_1 + m_2}, \quad q_r = \lambda_{12}(q_1 - q_2), \tag{A.9}$$

which leads to the c.m. and relative angles

$$\chi = \xi_1 + \xi_2, \quad \psi = \frac{m_2 \xi_1 - m_1 \xi_2}{\lambda_{12}(m_1 + m_2)}. \tag{A.10}$$

In terms of new variables, the two body Lagrangian becomes a sum of L_{cm} and L_{rel},

$$L_{\text{cm}} = \frac{1}{2}(m_1 + m_2)\dot{\mathbf{R}}^2 + \frac{1}{2(m_1 + m_2)}\dot{\chi}^2, \tag{A.11}$$

$$L_{\text{rel}} = \frac{1}{2}\left(\mu + \frac{\lambda_{12}}{r}\right)\dot{\mathbf{r}}^2 + \frac{\lambda_{12}^2}{2}\left(\mu + \frac{\lambda_{12}}{r}\right)^{-1}(\dot{\psi} + \mathbf{w}(\mathbf{r}) \cdot \dot{\mathbf{r}})^2 \tag{A.12}$$

The requirement that relative moduli space of two space being nonsingular is that the coupling constant λ_{12} should be positive and the relative coordinate ψ has to have a period of 4π. Let us consider the range of the angle parameters. The shift of ξ_1 by $4\pi t_1$ implies the identification

$$(\chi, \psi) = \left(\chi + 4\pi t_1, \psi + \frac{4\pi m_2 t_1}{\lambda_{12}(m_1 + m_2)}\right), \tag{A.13}$$

and the shift of ξ_2 by $-4\pi t_2$ implies the identification

$$(\chi, \psi) = \left(\chi - 4\pi t_2, \psi + \frac{4\pi m_1 t_2}{\lambda_{12}(m_1 + m_2)} \right) . \tag{A.14}$$

A combination of λ_{12}/t_1 steps of the first shift and λ_{12}/t_2 steps of the second shift will lead to an identification

$$(\chi, \psi) = (\chi, \psi + 4\pi) . \tag{A.15}$$

For this operation to be minimum so that the period of ψ is 4π, rather than a smaller number, λ_{12}/t_1 and λ_{12}/t_2 should be co-prime integers. There are several consequences from this requirement. The quantization of charge q_i leads to the relative charge as

$$q_{\rm rel} = \frac{\lambda_{12}}{2 t_1 t_2}(t_2 n_1 - t_1 n_2) , \tag{A.16}$$

with integer n_1, n_2. There are pair of integers such that $q_{\rm rel} = 1/2$ as expected. The ratio of the periods t_1/t_2 should be a positive rational number. After scaling the coordinates, we can make t_1, t_2 to be integers and λ_{12} to be the least common multiplet of t_1, t_2.

In short distance where $\mathbf{x}_1$ and $\mathbf{x}_2$ particles approach each other, the corresponding metric becomes

$$\mathcal{G}_{\rm rel} = \frac{\lambda_{12}}{2} ds_{R^4}^2 , \tag{A.17}$$

where

$$ds_{R^4}^2 = \frac{1}{r} d\mathbf{r}^2 + r(d\psi + \cos\theta d\phi)^2 \tag{A.18}$$

with the period of ψ in $[0, 4\pi]$ is the flat metric for Euclidean four dimensional space.

Generalizing to the N distinct monopoles of SU(N) gauge group, the ratio of any pair of periods of monopoles in adjacent point in the root diagram should be rational for the geometry to be nonsingular when two monopoles are coming together. As all monopoles are interacting each other at least indirectly, one can scale space and time and so the periods t_i are all integers, without any common factor. In our case also one can argue that the moduli space is smooth when $N - 1$ distinct monopoles are coming together by the argument similar to ref. [26]. However, the space becomes singular when N distinct monopoles coming together as they form a generalization of the moduli space of a single caloron of SU(N) gauge group [27]. Our generalization of the monopole moduli space for SU(3) would be exactly the three parameter (t_1, t_2, t_3) generalization of the moduli space $N(1, 1)$ of single SU(3) instanton. In the massless limit where mass of two monopoles vanishes, the monopole moduli space becomes $\mathcal{M}_{\mathbf{t}}$ after scaling of the coordinates.

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